# Tropical Geometry and Matroids

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# Overall Motivation

Tropical Geometry is an emerging field that enables certain algebraic geometric problems to be turned into combinatorial problems.
Already it has had striking applications to a wide range of subjects: [1]

- Enumerative geometry
- Classical geometry
- Intersection theory
- Moduli spaces and compactifications
- Mirror symmetry
- Abelian varieties
- Representation theory
- Algebraic statistics and mathematical biology

among other fields.

# Tropical Algebra

# The Three Semirings

The **tropical semiring** is one of three semirings:

1. The $(\min, +)$ semiring, which we will denote $\mathbb{T}_{\min}$. $\mathbb{T}_{\min}$ is the set $\mathbb{R} \cup \{\infty\}$ with:

$$a \boxplus b = \min\{a, b\} \qquad 3 \boxplus 7 = 3$$

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3. The $(\max, *)$ semiring, which we will denote $\mathbb{T}_{\times}$. $\mathbb{T}_{\times}$ is the set $\mathbb{R}_{\geq 0}$ with:

$$a \boxplus b = \max\{a, b\} \qquad 3 \boxplus 7 = 7$$
$$a \boxtimes b = a * b \qquad 3 \boxtimes 7 = 21$$

# Isomorphisms of $\mathbb{T}$

As semirings these are all isomorphic.

- $\mathbb{T}_{\text{min}} \cong \mathbb{T}_{\text{max}}$ via the isomorphism:

$$x \mapsto -x$$

- $\mathbb{T}_{\text{max}} \cong \mathbb{T}_{\times}$ via the isomorphism:

$$x \mapsto e^x$$

# Tropical Matrices and Applications

Tropical linear algebra has natural applications in combinatorial problems.

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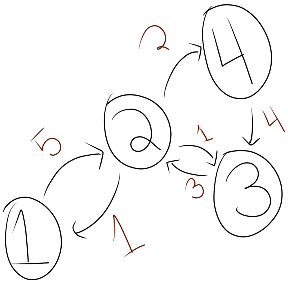
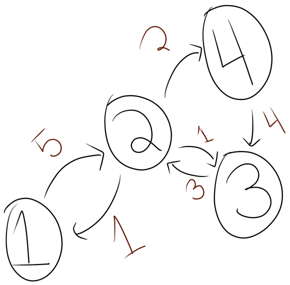
If you have a weighted graph:



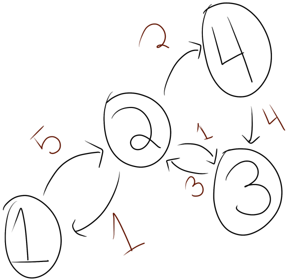
# Tropical Matrices and Applications

Tropical linear algebra has natural applications in combinatorial problems. You can turn it into an adjacency matrix:




$$M = \begin{bmatrix} 0 & 5 & \infty & \infty \\ 1 & 0 & 1 & 2 \\ \infty & 3 & 0 & \infty \\ \infty & \infty & 4 & 0 \end{bmatrix}$$

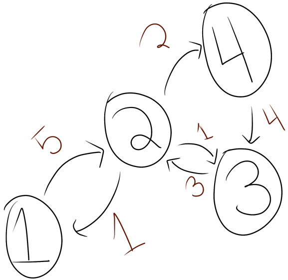
# Tropical Matrices and Applications




$$M = \begin{bmatrix} 0 & 5 & \infty & \infty \\ 1 & 0 & 1 & 2 \\ \infty & 3 & 0 & \infty \\ \infty & \infty & 4 & 0 \end{bmatrix}$$

When viewed as a matrix with entries in $\mathbb{T}_{\text{min}}$, $M$ has the property that the entry $m_{ij}$ of $M^n$ is the length of the shortest path in $n$ or fewer steps between nodes $i$ and $j$

# Tropical Matrices and Applications




$$M = \begin{bmatrix} 0 & 5 & \infty & \infty \\ 1 & 0 & 1 & 2 \\ \infty & 3 & 0 & \infty \\ \infty & \infty & 4 & 0 \end{bmatrix}$$

As each pair of nodes $i, j$ has a shortest path, we know that $M^{\infty} = M^{k}$ for some $k$ finite.

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## Definition

Given a set of agents $A$ and tasks $T$, with cost for agent $a$ to do task $t$ as $c(a, t)$. If one must do as many tasks as possible, and at most one agent can do each task, and each agent can do at most one task, what is the minimal possible cost?

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This is an important problem in computer science, as most scheduling problems can be reduced to an instance of the assignment problem.

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If we create a tropical matrix $M$ with a number of rows indexed by $A$ and columns indexed by $T$, and entries $m_{ij} = c(i,j)$ then the **Tropical Permanent** is the solution to the assignment problem.

$$\text{perm}(M) = \sum_{\sigma \in S_{|T|}} \prod_{i=1}^{|A|} m_{i,\sigma(i)}$$

Where all operations are viewed as operations in $\mathbb{T}_{\text{min}}$

# Tropical Geometry

# Fields with Valuation

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The general setting for Tropical Geometry is in the study of a **Field with Valuation**
Let $K$ be a field and $K^\times = K \setminus \{0\}$ its multiplicative group.

## Definition

A **valuation** on $K$ is a function:

$$\text{val} : K \to \mathbb{R} \cup \{\infty\}$$

Such that:

1. $\text{val}(a) = \infty$ if and only if $a = 0$
2. $\text{val}(ab) = \text{val}(a) + \text{val}(b)$
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This looks an awful lot like a semiring homomorphism $K \to \mathbb{T}_{\min}$, we would just need to strengthen condition (3) to be equality.

# Fields with Valuation

We can't always strengthen condition (3), however a standard result in the theory of valuations is the following:

**Theorem**

*If* $\text{val}(a) \neq \text{val}(b)$ *then* $\text{val}(a+b) = \min\{\text{val}(a), \text{val}(b)\}$

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**Corollary**

*The injective valuations* $K \to \mathbb{R} \cup \{\infty\}$ *are exactly the injective semiring homomorphisms* $K \to \mathbb{T}_{\text{min}}$

# Fields with Valuation

The best example to keep in mind is $\mathbb{Q}$ with the $p$-adic valuation for any prime $p$:

$$\operatorname{val}_p\left(p^k \frac{a}{b}\right) = k$$

Where $p \not| a, p \not| b$

# Fields with Valuation

Given a polynomial in a field with valuation $(K, \text{val})$

$$f = \sum a_u x^u \in K[x_1, ..., x_n]$$

We can define the tropicalization of $f$ as:

$$\text{trop}(f) = \min(\text{val}(a_u) + u_1 x_1 + ... u_n x_n) \in \mathbb{T}_{\min}[x_1, ..., x_n]$$

# Method of Newton Polygons

One of the earliest examples of Tropical Geometry comes from a 1676 letter from Isaac Newton to Henry Oldenburg

# Method of Newton Polygons

Let $K$ be a field with valuation and let:

$$f(x) = a_n x^n + \ldots + a_1 x + a_0 \in K[x]$$

With $a_n a_0 \neq 0$
We define the Newton Polygon to be the lower convex hull of the set of points:

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**Theorem**

*Let $\mu_1, \ldots, \mu_r$ be the slopes of the line sements of the Newton polygon of $f(x)$ with $\lambda_1, \ldots, \lambda_r$ the corresponding lengths of the projections of those lines onto the $x$-axis. Then for each $1 \leq i \leq r$, $f(x)$ has exactly $\lambda_i$ roots in its splitting field with valuation $-\mu_i$*

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Consider the polynomial

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The Newton Polygon is the lower convex hull of the set:

$$\{(0,5),(1,2),(2,0),(3,0)\}$$

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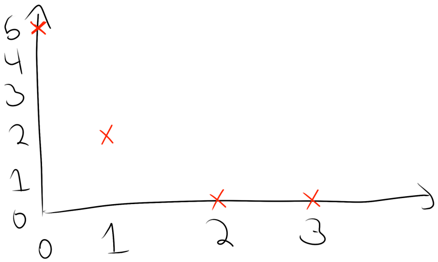

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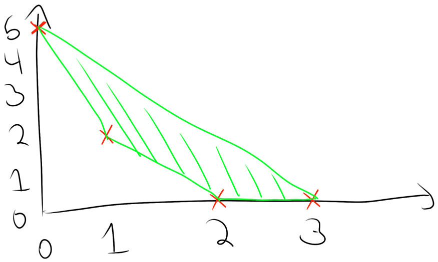

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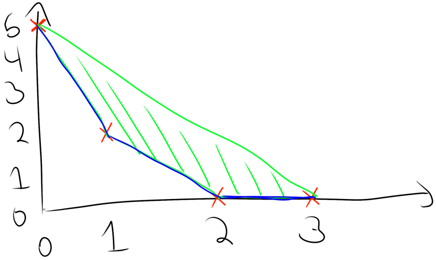

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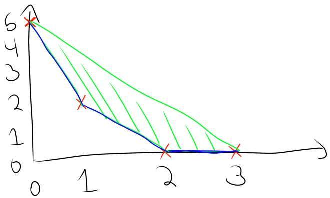


So we have slopes $-3, -2, 0$ with corresponding lengths $1, 1, 1$ – so we have a single root of valuation 3, a single root of valuation 2, and a single root of valuation 0.

# Tropical Varieties

Let $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$
The classical variety is a hypersurface in the algebraic torus $T^n = (\bar{K}^\times)^n$ over the algebraic closure of $K$:

$$V(f) = \{y \in T^n : f(y) = 0\}$$

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The **tropical hypersurface** $\mathrm{trop}(V(f))$ is the set:

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This gives us:

$$\mathrm{trop}(V(f)) = V(\mathrm{trop}(f))$$

# Tropical Varieties

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The **tropicalization of** $X$ is:

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We call a finite generating set $\mathcal{T} \subset I$ a *tropical basis* of $I$ if:

$$\operatorname{trop}(V(I)) = \bigcap_{f \in \mathcal{T}} \operatorname{trop}(V(f))$$

It can be shown that every Laurent ideal has a finite tropical basis [2][Theorem 2.6.6]

# Fundamental Theorem of Tropical Algebraic Geometry

**Theorem**

*Let $K$ be an algebraically closed field with nontrivial valuation. Let $I$ be an ideal in $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and let $X = V(I)$ be its variety in the algebraic torus $T^n \cong (K^\times)^n$. The following subsets of $\mathbb{R}^n$ coincide:*

1. *The tropical variety* $\mathrm{trop}(X)$
2. *The set of all vectors* $\mathrm{w} \in \mathbb{R}^n$ *with* $\mathrm{in}_{\mathrm{w}}(I) \neq \langle 1 \rangle$
3. *The closure of the set of coordinatewise valuations of points in $X$:*

$$\mathrm{val}(X) = \{(\mathrm{val}(y_1), \dots, \mathrm{val}(y_n) : (y_1, \dots, y_n) \in X\}$$

*Furthermore if $X$ is irreducible and* $\mathrm{w}$ *is any point in* $\Gamma^n_{\mathrm{val}} \cap \mathrm{trop}(X)$ *then the set* $\{\mathrm{y} \in X : \mathrm{val}(\mathrm{y}) = \mathrm{w}\}$ *is Zariski dense in $X$.*

# Structure Theorem for Tropical Varieties

**Theorem**

*Let $X$ be an irreducible $d$-dimensional subvariety of $T^n$. Then $\mathrm{trop}(X)$ is the support of a balanced weighted $\Gamma_{\mathrm{val}}$-rational polyhedral complex pure of dimension $d$. Moreover that polyhedral complex is connected through codimension $1$.*

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- **Pure of dimension** $d$: All maximal cells have dimension $d$
- **Balanced, weighted:** For each $d-1$ dimensional cell, the sum of the weighted vectors to the first lattice point generator of each $d$ dimensional cell containing it are zero. The weights for our tropical variety are the lattice lengths of the dual polyhedral complex. (Tropical varieties are dual to regular subdivisions of Newton polytopes)
- **Connected through codimension 1:** For any pair of $d$ dimensional (maximal) cells you can find a path connecting them through $d-1$ dimensional cells.

# Structure Theorem for Tropical Varieties

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This theorem has a partial converse. If $d = n - 1$ then any balanced weighted $\Gamma_{\mathrm{val}}$-rational polyhedral complex pure of dimension $d$ which is connected through codimension 1, $\Sigma$, then there is some variety $X \subset T^n$ such that $\Sigma = \mathrm{trop}(X)$.

# Linear Spaces and Matroids

# No Subtraction Adds Problems

This is all well and good if we are starting with a field, and trying to answer problems about its geometry, but what if we want to look at purely tropical objects?

# No Subtraction Adds Problems

Reminder, given an ideal $K[x_1, \ldots, x_n] \supset I = \langle f_1, \ldots, f_m \rangle$, the associated tropical variety is:

$$\bigcap_{f \in I} V(\mathrm{trop}(f))$$

We can also phrase this as the $V(\mathrm{trop}(I))$, where $\mathrm{trop}(I)$ is the ideal in $\mathbb{T}[x_1, \ldots, x_n]$ with:

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This is *almost always* a strict inclusion
Consider: $\text{trop}(\langle x + y, x - y \rangle)$. This contains $x$, however if we look at the ideal in the tropical semiring generated by the generators we get:

$$\langle x \oplus y \rangle$$

Which does not contain $x$.

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Because of this, arbitrary ideals of the tropical semiring do not follow the fundamental theorem of tropical geometry. Their varieties can be arbitrary, non-convex structures.

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Because of this, arbitrary ideals of the tropical semiring do not follow the fundamental theorem of tropical geometry. Their varieties can be arbitrary, non-convex structures. If we want to take a “tropical first" approach to tropical geometry, we need an object that behaves like the tropicalization of an ideal even if it is not.

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Linearity gives us a concept of *elimination* or subtraction. So to solve the problems that the lack of subtraction gives, we will just require each degree $d$ component of our tropical ideal to behave like the tropicalization of a linear space.

# Tropical Linear Spaces

**Definition**

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Tropical Ideals, even when they are not the tropicalizations of an ideal, follow both the structure theorem and the fundamental theorem.

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A tropical linear space is a tropical ideal whose structure is given by a *matroid*.

# Matroids

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In each of these examples dependency captures some sort of redundancy or elimination that is happening.

# Matroids

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**Definition (Independent Set)**

A finite **matroid** $M$ is a pair $(E, I)$ where $E$ is a finite set called the **ground set** and $I \subset 2^E$ is a family of subsets of $E$ called the **independent sets** which follow the following axioms:

1. $\emptyset \in I$
2. Every subset of an independent set is independent.
3. **Independent Set Exchange Axiom:** If $A, B$ are two independent sets and $|A| > |B|$ then there is some $a \in A \setminus B$ such that $B \cup \{a\} \in I$.

The first two axioms give the definition of an *independence system* or an *abstract simplicial complex*, the third defines the matroid.

# Matroids

Instead of the full independence system, we only need the *maximal independent sets* to describe the matroid, we call those **bases**:

## Definition (Bases)

A finite **matroid** $M$ is a finite set $E$ and a nonempty collection of subsets of $E$, $\mathcal{B}$ called the **bases** of $M$ such that:

1. No proper subset of an element of $\mathcal{B}$ is in $\mathcal{B}$
2. **Basis Exchange Axiom:** If $A, B$ are two distinct members of $\mathcal{B}$ and $a \in A \setminus B$ then there exists $b \in B \setminus A$ such that $(A \setminus \{a\}) \cup \{b\} \in \mathcal{B}$

The second axiom is the same as the basis exchange theorems in Linear Algebra.

# Matroids

Instead of the full independence system, we only need the *minimal dependent sets* to describe the matroid, we call those **circuits**:

## Definition (Circuits)

A finite **matroid** $M$ is a finite set $E$ and a nonempty collection of subsets of $E$, $\mathcal{C}$ called the **circuits** of $M$ such that:

1. No proper subset of an element of $\mathcal{C}$ is in $\mathcal{C}$
2. **Circuit Exchange Axiom:** If $A, B$ are two distinct members of $\mathcal{C}$ and $c \in A \cap B$ then $(A \cup B) \setminus \{c\}$ contains some circuit.

# Matroids

Each of those definitions follows pretty closely from one another, but not all definitions of matroids even seem to have *anything* to do with an independence system.

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## Definition (Rank)

A finite **matroid** $M$ is a finite set $E$ with a **rank function** $r : 2^E \to \mathbb{Z}_+$ such that:

1. Rank is at most the size of your set: $r(A) \leq |A|$
2. The rank function is submodular: $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$
3. Rank is monotonic: For any $A \subset E$ and $x \in E$, $r(A) \leq r(A \cup \{x\}) \leq r(A) + 1$

Rank can be thought of as the *dimension* of a substructure generated by the given subset.

# Matroids

The closure of a set $A \subset E$ is the set:

$$\mathrm{cl}(A) = \{x \in E : r(A) = r(A \cup \{x\})$$

Knowing these closures alone is enough to recreate the matroid. A set $A$ such that $\mathrm{cl}(A) = A$ is called a **flat** of a matroid.

# Matroids

**Definition (Lattice of Flats)**

A finite **matroid** $M$ is a finite set $E$ and a collection of subsets of $E$, $\mathcal{F} \subset 2^E$ called the **flats** of $M$ such that:

1. $E$ is itself a flat.
2. Flats are closed under intersection: If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$
3. The flats which cover $A$ partition $E \setminus A$: If $A$ is a flat, then each element of $E \setminus A$ is in precisely one of the flats $T$ that cover $A$ – a set $T$ covers $A$ if $A \subset T$ and there are no flats $X$ such that $A \subset X \subset T$

The collection of flats forms a lattice under inclusion.

# Matroid Examples

**Vector Spaces**: Given a vector space $V$ and a finite collection of vectors $E \subset V$ we get a matroid by:

- $S \subset E$ is independent if it is a linearly independent set
- $S \subset E$ is a basis if it forms a linear basis of $\text{span}(E)$
- $S \subset E$ is a circuit if the dimension of its span is $|S| - 1$
- $r(S) = \dim(\text{span}(S))$
- $S$ is a flat if there is no $x \in E \setminus S$ such that $x \in \text{span}(S)$

Such matroids are called **representable** matroids. If that matroid structure can be formed over any possible field it is called a **regular** matroid

# Matroid Examples

**Graphs**: Given a (multi)graph $G = (V, E)$:

- $S \subset E$ is independent if it does not contain a cycle
- $S \subset E$ is a basis if it forms a minimal spanning forest of $G$
- $S \subset E$ is a circuit if it is a simple cycle
- $r(S) = n - c$ where $n$ is the number of vertices in the subgraph determined by $S$ and $c$ is the number of connected components.
- The flats of $G$ are partitions of $G$ into connected components

Such matroids are called **graphic** matroids.

# Matroid Examples

**Field Extensions**: Let $K$ be a field extension of $F$ and let $E$ be a finite subset of $K$:

- $S \subset E$ is independent if the extension field $F(S)$ has transcendence degree over $F$ equal to $|S|$
- $r(S)$ is the transcendence degree of $F(S)$ over $S$

Such matroids are called **algebraic** matroids.

# Matroid Examples

There are a hierarchy of matroids.
Over fields of characteristic zero algebraic matroids are representable matroids, but in general they form a larger class.

$$\text{graphic} \subset \text{regular} \subset \text{representable} \subset \text{algebraic} \subset \text{matroids}$$

With every single one of those inclusions being strict.

# Why are matroids cool in general?

Have you ever wondered when greedy algorithms are optimal, versus when they are just approximations?
The answer is Matroids!

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**Theorem**

*A greedy algorithm is optimal if and only if it can be formulated as an algorithm over a matroid in the following way:*
*Given a matroid* $M = (E, I)$ *with a cost function* $c : E \to \mathbb{R}^+$*, the greedy algorithm iteratively adds the cheapest element of* $E$ *so long as it remains independent.*

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In the same way that path algebras are the “algebraic representation" of the minimal cost problem, matroids are the algebraic representation of greedy algorithms.

# Thank you for attending!

Questions?

# References

# References I

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